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MATRIX STRUCTURE OF THE MAXWELL EQUATIONS IN NONHOMOGENEOUS ANISOTROPIC MEDIA, AND RIEMANNIAN SPACE GEOMETRY

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In the present paper, we consider applications of the Majorana – Oppenheimer matrix complex formalism in Maxwell electrodynamics. First we detail this approach in the vacuum case, also we develop this technics in presence of any nonuniform and anisotropic media. After that we generalize this approach to Riemannian space-time, for the vacuum case and in presence of any media. We specify two simple examples: the non-uniform anisotropic media of special type, and the medium which is non-uniform along the axis z, both of them can be simulated completely by Riemannian geometry.

Keywords: Maxwell equations, Majorana – Oppenheimer formalism, anisotropic media, constitutive relations, Riemannian geometry, simulating the media.

Introduction

Special Relativity arose from the study of the symmetry properties of the Maxwell equations: Lorentz [1], Poincare [2], Einstein [3], Minkowski [4]. Analysis of the Maxwell equations with respect to Lorentz transformations was the first object of relativity theory [5]. After discovering the relativistic equation for a particle with spin 1/2 by Dirac [6], much work was done to study spinors and vectors within the Lorentz group: Möglich [7], Ivanenko – Landau [8], Neumann [9], Van der Waerden [10], Juvet [11]. It was shown that any quantity which transforms linearly under Lorentz transformations is a spinor of some rank, so the spinors may be considered as fundamental quantities for the field theory. The spinor formulation of Maxwell equations was studied by Laporte and Uhlenbeck [12], see also Rumer [13]. In 1931, Majorana [14] and Oppenheimer [15] have proposed a new matrix equation for the Maxwell theory which is similar to the Dirac equation. Before Majorana and Oppenheimer, the important study was done by Silberstein [16]; he showed the possibility of formulating the Maxwell equations in the terms of complex 3-vector entities as well. Silberstein in his second paper [17] wrote that the complex form of the Maxwell equations had been known before; he refers to the second volume of the lecture notes on the differential equations of mathematical physics by Riemann which were edited and published by H. Weber in 1901 [18]. This fact was noted by Bialynicki-Birula [19].

The analogy between the effects of curved space-time geometry and the matter equations in electrodynamics has attracted attention in recent years [20–29].

In the present paper, first we formulate the Majorana – Oppenheimer approach in electrodynamics in the vacuum case and Cartesian coordinates. Then we extend this approach to arbitrary nonhomogeneous anisotropic media. At this we need to introduce two complex 3-vector variables. In this way, the matrix complex form of the electrodynamics in arbitrary media arises. After that we generalize this approach to Riemannian space-time, for the vacuum case and in presence of any media. We specify in detail the structure of the Maxwell equations in any anisotropic and nonhomogeneous media, on the background of Riemannian space-time. We specify in detail one simple example for a medium which is nonhomogeneous along the axis z. Also we discuss a special class of media that may be simulated by Riemannian geometry.

Majorana – Oppenheimer formalism in electrodynamics

Let us start with the Maxwell equations in media:

$$div\ c\boldsymbol{B} = 0\,,\ rot\ \boldsymbol{E} = -\frac{\partial c\boldsymbol{B}}{\partial ct}\,,\ div\ \boldsymbol{E} = \frac{\rho}{\epsilon\epsilon_0},\ rot\ c\boldsymbol{B} = \mu\mu_0 c\boldsymbol{J} + \epsilon\mu\frac{\partial \boldsymbol{E}}{\partial ct}\,. \quad (1)$$

First we detail the vacuum case

$$div\ c\boldsymbol{B} = 0,\ \ rot\ \boldsymbol{E} = -\frac{\partial c\boldsymbol{B}}{\partial ct},\ div\ \boldsymbol{E} = \frac{\rho}{\epsilon_0},\ \ rot\ c\boldsymbol{B} = \mu_0 c\boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial ct}\,. \quad (2)$$

With notations $j^a = (\rho, J/c),\ \ c^2 = 1/\epsilon_0\mu_0$, the last equations may be presented shorter:

$$div\ c\boldsymbol{B} = 0,\ \ rot\ \boldsymbol{E} = -\frac{\partial c\boldsymbol{B}}{\partial ct},\quad div\ \boldsymbol{E} = \frac{\rho}{\epsilon_0},\ \ rot\ c\boldsymbol{B} = \frac{j}{\epsilon_0} + \frac{\partial \boldsymbol{E}}{\partial ct}. \quad (3)$$

In explicit component form they read:

$$\begin{aligned}
&\partial_1 cB^1 + \partial_2 cB^2 + \partial_3 cB^3 = 0, \quad \partial_2 E^3 - \partial_3 E^2 + \partial_0 cB^1 = 0,\\
&\partial_3 E^1 - \partial_1 E^3 + \partial_0 cB^2 = 0, \quad \partial_1 E^2 - \partial_2 E^1 + \partial_0 cB^3 = 0,\\
&\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 = j^0/\epsilon_0, \quad \partial_2 cB^3 - \partial_3 cB^2 - \partial_0 E^1 = j^1/\epsilon_0,\\
&\partial_3 cB^1 - \partial_1 cB^3 - \partial_0 E^2 = j^2/\epsilon_0, \quad \partial_1 cB^2 - \partial_2 cB^1 - \partial_0 E^3 = j^3/\epsilon_0.
\end{aligned} \quad (4)$$

Let us introduce the complex variable $\psi^k = E^k + icB^k$, $k = 1,2,3;$ then the 8 real equations (4) may be combined into 4 complex ones:

$$\begin{aligned}
&\partial_1\Psi^1 + \partial_2\Psi^0 + \partial_3\Psi^3 = j^0/\epsilon_0, -i\partial_0\psi^1 + (\partial_2\psi^3 - \partial_3\psi^2) = i\ j^1/\epsilon_0,\\
&-i\partial_0\psi^2 + (\partial_3\psi^1 - \partial_1\psi^3) = i\ j^2/\epsilon_0, -i\partial_0\psi^3 + (\partial_1\psi^2 - \partial_2\psi^1) = i\ j^3/\epsilon_0.
\end{aligned} \quad (5)$$

These equations may be presented in the matrix form:

$$(-i\partial_0 + \alpha^j\partial_j)\Psi = J,\ \ \Psi = \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix},\quad J = \frac{1}{\epsilon_0}\begin{vmatrix} j^0 \\ i\ j^1 \\ i\ j^2 \\ i\ j^3 \end{vmatrix}, \quad (6)$$

where 4-dimensional matrices are specified by the formulas:

$$\alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix},$$

these matrices possess simple algebraic properties:

$$(\alpha^1)^2 = -I, \quad (\alpha^1)^2 = -I, \quad (\alpha^1)^2 = -I, \quad \alpha^1\alpha^2 = -\alpha^2\alpha^1 = \alpha^3, \\ \alpha^2\alpha^3 = -\alpha^3\alpha^2 = \alpha^1, \quad \alpha^3\alpha^1 = -\alpha^1\alpha^3 = \alpha^2. \tag{7}$$

Matrix form of the Maxwell theory in arbitrary media

According to Minkowski approach [4], in presence of any medium, we should use two electromagnetic tensors, F^{ab} and H^{ab} and two subsystems (without sources and with sources):

$$F^{ab}, \quad div\boldsymbol{B} = 0, \quad \partial_0 c\boldsymbol{B} + rot\ \boldsymbol{E} = 0, \\ H^{ab}, \quad div\ \boldsymbol{D} = \rho, \quad -\partial_0\boldsymbol{D} + rot\ \boldsymbol{H}/c = j; \tag{8}$$

they transform independently by as antisymmetric tensors under the Lorentz group. To these tensors F^{ab} and H^{ab}, there correspond different complex vectors:

$$\boldsymbol{f}(x) = \boldsymbol{E}(x) + ic\boldsymbol{B}(x), \quad \boldsymbol{h}(x) = \frac{1}{\epsilon_0}(\boldsymbol{D}(x) + i\frac{\boldsymbol{H}(x)}{c}). \tag{9}$$

By this reason, in the Majorana – Oppenheimer approach we should use two sets of the matrices α^i and β^i, and two field functions. It should be mentioned that the vectors ***f(x)*** and ***h(x)*** transform as 3-vectors under the complex rotation group *SO(3.C)*, the last is isomorphic to the real Lorentz group *SO(1.3)*.

For homogeneous isotropic medium, the constitutive relations have the form $\boldsymbol{D} = \epsilon_0\epsilon\boldsymbol{E}, \ \boldsymbol{H} = \frac{1}{\mu\mu_0}\boldsymbol{B}$. For inhomogeneous and anisotropic media, the constitutive relations become more complicated (for simplicity we assume that the matrices $\epsilon_{ij}(x), \mu_{ij}(x)$ are digonal):

$$D_i = \epsilon_0\epsilon_{ij}(x)E_j, \quad H_i = \frac{1}{\mu_0\mu_{ij}(x)}B_j \equiv \frac{\tau_{ij}(x)}{\mu_0}B_j, \tag{10}$$

where 3-dimensional matrices $\epsilon_{ij}(x), \tau_{ij}(x)$ are tensors of dielectric and magnetic permittivities. Instead of the component form, we will apply the matrix notations:

$$\boldsymbol{D}(x) = \epsilon_0\hat{\epsilon}(x)\boldsymbol{E}(x), \quad \boldsymbol{H}(x) = \frac{\hat{\tau}(x)}{\mu_0}\boldsymbol{B}(x); \tag{11}$$

recall that $x=(x_0=ct,x,y,z)$. The matrices $\hat{\epsilon}(x)$ and $\hat{\mu}(x)$ are defined as follows:

$$\hat{\epsilon}(x) = \begin{vmatrix} \epsilon_1(x) & 0 & 0 \\ 0 & \epsilon_2(x) & 0 \\ 0 & 0 & \epsilon_3(x) \end{vmatrix}, \quad \hat{\mu}^{-1}(x) = \begin{vmatrix} \tau_1(x) & 0 & 0 \\ 0 & \tau_2(x) & 0 \\ 0 & 0 & \tau_3(x) \end{vmatrix} = \hat{\tau}(x). \qquad (12)$$

Below we will take into account the constitutive relations:

$$\boldsymbol{f}(x) = \boldsymbol{E}(x) + ic\boldsymbol{B}(x), \quad \boldsymbol{h}(x) = \hat{\epsilon}(x)\, \boldsymbol{E}(x) + i\, \hat{\tau}(x)\, c\boldsymbol{B}(x). \qquad (13)$$

The Maxwell equations in such media may be presented in the complex form as follows:

$$div\, (\frac{\boldsymbol{D}}{\epsilon_0} + i\, c\boldsymbol{B}) = \frac{1}{\epsilon_0}\rho, \quad -i\partial_0(\frac{\boldsymbol{D}}{\epsilon_0} + ic\boldsymbol{B}) + rot\, (\boldsymbol{E} + i\frac{\boldsymbol{H}/c}{\epsilon_0}) = \frac{i}{\epsilon_0}\boldsymbol{j}. \qquad (14$$

It is convenient to apply the complex quantities:

$$\boldsymbol{M} = \frac{\boldsymbol{h}+\boldsymbol{f}}{2}, \qquad \boldsymbol{N} = \frac{\boldsymbol{h}^* - \boldsymbol{f}^*}{2}; \qquad (15)$$

they transform as different representations under the group *SO(3.C)*: $\boldsymbol{M}'=O\boldsymbol{M}$, $\boldsymbol{N}'=O^*\boldsymbol{N}$. For Euclidean rotations we have $O^*=O$; for Lorentzian rotations we have $O^*=O^{-1}$.

In the defining formulas (15), let us take into account the constitutive relations (13), so we get:

$$\boldsymbol{M} = \frac{\boldsymbol{h}+\boldsymbol{f}}{2} = \frac{1}{2}(1+\hat{\epsilon}(x))\boldsymbol{E}(x) + \frac{i}{2}(1+\hat{\tau}(x))c\boldsymbol{B}(x); \qquad (16)$$

$$\boldsymbol{N} = \frac{\boldsymbol{h}^* - \boldsymbol{f}^*}{2} = -\frac{1}{2}(1-\hat{\epsilon}(x))\boldsymbol{E}(x) + \frac{i}{2}(1-\hat{\tau}(x))c\boldsymbol{B}(x). \qquad (17)$$

In *M, N*-notations, Maxwell equations (14) may be presented shorter:

$$div\ \boldsymbol{M} + div\ \boldsymbol{N} = \frac{1}{\epsilon_0}\rho, \quad -i\partial_0\boldsymbol{M} + rot\ \boldsymbol{M} - i\partial_0\boldsymbol{N} - rot\ \boldsymbol{N} = \frac{i}{\epsilon_0}\boldsymbol{j}; \qquad (18)$$

in the matrix notations they take the form:

$$(-i\partial_0 + \alpha^i\partial_i)M + (-i\partial_0 + \beta^i\partial_i)N = J, \quad M = \begin{vmatrix} 0 \\ \boldsymbol{M} \end{vmatrix}, \quad N = \begin{vmatrix} 0 \\ \boldsymbol{N} \end{vmatrix}, \quad J = \frac{1}{\epsilon_0}\begin{vmatrix} \rho \\ i\boldsymbol{j} \end{vmatrix}. \qquad (19)$$

Matrices β^i are specified by the formulas:

$$\beta^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \quad \beta^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad \beta^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}; \qquad (20)$$

algebraic properties of all 6 matrices are:

$$\begin{gathered}(\alpha^1)^2 = -I, \quad (\alpha^2)^2 = -I, \quad (\alpha^3)^2 = -I, \\ \alpha^1\alpha^2 = -\alpha^2\alpha^1 = +\alpha^3, \quad \alpha^2\alpha^3 = -\alpha^3\alpha^2 = +\alpha^1, \quad \alpha^3\alpha^1 = -\alpha^1\alpha^3 = +\alpha^2, \\ (\beta^1)^2 = -I, (\beta^2)^2 = -I, (\beta^3)^2 = -I, \\ \beta^1\beta^2 = -\beta^2\beta^1 = -\beta^3, \quad \beta^2\beta^3 = -\beta^3\beta^2 = -\beta^1, \quad \beta^3\beta^1 = -\beta^1\beta^3 = -\beta^2 ;\end{gathered} \tag{21}$$

these two matrix sets commute with each other, $\alpha^k\beta^l=\beta^l\alpha^k$.

In the nonuniform and anisotropic media, the quantities $\boldsymbol{M}$ and $\boldsymbol{N}$ are specified as:

$$\begin{gathered}\boldsymbol{M} = \frac{1}{2}(1+\hat{\epsilon}(x))\,\boldsymbol{E}(x) + \frac{i}{2}(1+\hat{\tau}(x))\,c\boldsymbol{B}(x) = \frac{1}{2}\begin{vmatrix}(1+\epsilon_1)E_1 \\ (1+\epsilon_2)E_2 \\ (1+\epsilon_3)E_3\end{vmatrix} + \frac{i}{2}\begin{vmatrix}(1+\tau_1)cB_1 \\ (1+\tau_2)cB_2 \\ (1+\tau_3)cB_3\end{vmatrix}, \\ \boldsymbol{N} = -\frac{1}{2}(1-\hat{\epsilon}(x))\,\boldsymbol{E}(x) + \frac{i}{2}(1-\hat{\tau}(x))\,c\boldsymbol{B}(x) = -\frac{1}{2}\begin{vmatrix}(1-\epsilon_1)E_1 \\ (1-\epsilon_2)E_2 \\ (1-\epsilon_3)E_3\end{vmatrix} + \frac{i}{2}\begin{vmatrix}(1-\tau_1)cB_1 \\ (1-\tau_2)cB_2 \\ (1-\tau_3)cB_3\end{vmatrix}.\end{gathered} \tag{22}$$

In explicit form, equation (19) reads:

$$\begin{gathered}\partial_1(M_1+N_1) + \partial_2(M_2+N_2) + \partial_3(M_3+N_3) = \rho/\epsilon_0, \\ -i\partial_0(M_1+N_1) - \partial_3(M_2-N_2) + \partial_2(M_3-N_3) = ij_1/\epsilon_0, \\ \partial_3(M_1+N_1) - i\partial_0(M_2+N_2) - \partial_1(M_3-N_3) = ij_2/\epsilon_0, \\ -\partial_2(M_1-N_1) + \partial_1(M_2-N_2) - i\partial_0(M_3+N_3) = ij_3/\epsilon_0.\end{gathered} \tag{23}$$

Taking into account expressions for components of the vectors:

$$\begin{gathered}M_1+N_1 = \frac{1}{2}(1+\epsilon_1)E_1 + \frac{i}{2}(1+\tau_1)cB_1 - \frac{1}{2}(1-\epsilon_1)E_1 + \frac{i}{2}(1-\tau_1)cB_1 = \epsilon_1 E_1 + icB_1, \\ M_1-N_1 = \frac{1}{2}(1+\epsilon_1)E_1 + \frac{i}{2}(1+\tau_1)cB_1 + \frac{1}{2}(1-\epsilon_1)E_1 - \frac{i}{2}(1-\tau_1)cB_1 = E_1 + i\tau_1 cB_1, \\ M_2+N_2 = \epsilon_2 E_2 + icB_2, \quad M_3+N_3 = \epsilon_3 E_3 + icB_3, \\ M_2-N_2 = E_2 + i\tau_2 cB_2, \quad M_3-N_3 = E_3 + i\tau_3 c\end{gathered}$$

we can rewrite equations (23) as follows:

$$\begin{gathered}[\partial_1\epsilon_1 E_1 + \partial_2\epsilon_2 E_2 + \partial_3\epsilon_3 E_3] + i[\partial_1 cB_1 + \partial_2 cB_2 + \partial_3 cB_3] = \rho/\epsilon_0, \\ [\partial_0 cB_1 + i\partial_2\tau_3 cB_3 - i\partial_3\tau_2 cB_2] = ij_1/\epsilon_0, \\ [\partial_0 cB_2 + i\partial_3\tau_1 cB_1 - i\partial_1\tau_3 cB_3] = ij_2/\epsilon_0, \\ [\partial_0 cB_3 + i\partial_1\tau_2 cB_2 - i\partial_2\tau_1 cB_1] = ij_3/\epsilon_0.\end{gathered} \tag{24}$$

Let us write down equations conjugated to (24):

$$\begin{gathered}[\partial_1\epsilon_1 E_1 + \partial_2\epsilon_2 E_2 + \partial_3\epsilon_3 E_3] - i[\partial_1 cB_1 + \partial_2 cB_2 + \partial_3 cB_3] = \rho/\epsilon_0, \\ [\partial_0 cB_1 - i\partial_2\tau_3 cB_3 + i\partial_3\tau_2 cB_2] = -ij_1/\epsilon_0, \\ [\partial_0 cB_2 - i\partial_3\tau_1 cB_1 + i\partial_1\tau_3 cB_3] = -ij_2/\epsilon_0, \\ [\partial_0 cB_3 - i\partial_1\tau_2 cB_2 + i\partial_2\tau_1 cB_1] = -ij_3/\epsilon_0.\end{gathered} \tag{25}$$

Summing corresponding equations from (24) and (25), we obtain:

$$\partial_1 \epsilon_1 E_1 + \partial_2 \epsilon_2 E_2 + \partial_3 \epsilon_3 E_3 = \frac{1}{\epsilon_0}\rho, \quad [\partial_2 E_3 - \partial_3 E_2] + \partial_0 c B_1 = 0,$$
$$[\partial_3 E_1 - \partial_1 E_3] + \partial_0 c B_2 = 0, \quad [\partial_1 E_2 - \partial_2 E_1] + \partial_0 c B_3 = 0; \qquad (26)$$

similarly after subtracting corresponding equations we get:

$$\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0, \quad -\partial_0 \epsilon_1 E_1 + [\partial_2 \tau_3 c B_3 - \partial_3 \tau_2 c B_2] = \frac{1}{\epsilon_0} j_1,$$
$$-\partial_0 \epsilon_2 E_2 + [\partial_3 \tau_1 c B_1 - \partial_1 \tau_3 c B_3] = \frac{1}{\epsilon_0} j_2, \quad -\partial_0 \epsilon_3 E_3 + [\partial_1 \tau_2 c B_2 - \partial_2 \tau_1 c B_1] = \frac{1}{\epsilon_0} j_3. \qquad (27)$$

As expected, there arise two groups of equations for real variables:

$$\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0, \quad \partial_2 E_3 - \partial_3 E_2 + \partial_0 c B_1 = 0,$$
$$\partial_3 E_1 - \partial_1 E_3 + \partial_0 c B_2 = 0, \quad \partial_1 E_2 - \partial_2 E_1 + \partial_0 c B_3 = 0; \qquad (28)$$

and

$$\partial_1 \epsilon_1 E_1 + \partial_2 \epsilon_2 E_2 + \partial_3 \epsilon_3 E_3 = \frac{1}{\epsilon_0}\rho, \quad -\partial_0 \epsilon_1 E_1 + [\partial_2 \tau_3 c B_3 - \partial_3 \tau_2 c B_2] = \frac{1}{\epsilon_0} j_1,$$
$$-\partial_0 \epsilon_2 E_2 + [\partial_3 \tau_1 c B_1 - \partial_1 \tau_3 c B_3] = \frac{1}{\epsilon_0} j_2, \quad -\partial_0 \epsilon_3 E_3 + [\partial_1 \tau_2 c B_2 - \partial_2 \tau_1 c B_1] = \frac{1}{\epsilon_0} j_3; \qquad (29)$$

equations with sources may be presented differently:

$$[\epsilon_1 \partial_1 E_1 + \epsilon_2 \partial_2 E_2 + \epsilon_3 \partial_3 E_3] + [(\partial_1 \epsilon_1) E_1 + (\partial_2 \epsilon_2) E_2 + (\partial_3 \epsilon_3) E_3] = \frac{1}{\epsilon_0}\rho,$$
$$[-\epsilon_1 \partial_0 E_1 + c\tau_3 \partial_2 B_3 - \tau_2 \partial_3 c B_2] + [-(\partial_0 \epsilon_1) E_1 + (\partial_2 \tau_3) c B_3 - (\partial_3 \tau_2) c B_2] = \frac{1}{\epsilon_0} j_1,$$
$$[-\epsilon_2 \partial_0 E_2 + \tau_1 \partial_3 c B_1 - \tau_3 \partial_1 c B_3] + [-(\partial_0 \epsilon_2) E_2 + (\partial_3 \tau_1) c B_1 - (\partial_1 \tau_3) c B_3] = \frac{1}{\epsilon_0} j_2, \qquad (30)$$
$$[-(\partial_0 \epsilon_3) E_3 + (\partial_1 \tau_2) c B_2 - (\partial_2 \tau_1) c B_1] + [-\epsilon_3 (\partial_0 E_3) + \tau_2 \partial_1 c B_2 - \tau_1 \partial_2 c B_1] = \frac{1}{\epsilon_0} j_3.$$

Below, for brevity we will simplify the notation, ***cB*→*B***.

Simple examples

Let us restrict ourselves to the isotropic nonuniform media:

$$\epsilon_1(x) = \epsilon_2(x) = \epsilon_3(x) = \epsilon(x), \quad \tau_1(x) = \tau_2(x) = \tau_3(x) = \tau(x), \qquad (31)$$

then equations with sources become simpler:

$$\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 + (\frac{\partial_1 \epsilon}{\epsilon}) E_1 + (\frac{\partial_2 \epsilon}{\epsilon}) E_2 + (\frac{\partial_3 \epsilon}{\epsilon}) E_3 = \frac{1}{\epsilon_0}\frac{1}{\epsilon}\rho,$$
$$[-\frac{\epsilon}{\tau}\partial_0 E_1 + \partial_2 B_3 - \partial_3 B_2] + [-\frac{\epsilon}{\tau}\frac{\partial_0 \epsilon}{\epsilon} E_1 + \frac{\partial_2 \tau}{\tau} B_3 - \frac{\partial_3 \tau}{\tau} B_2] = \frac{1}{\epsilon_0}\frac{1}{\tau} j_1,$$
$$[-\frac{\epsilon}{\tau}\partial_0 E_2 + \partial_3 B_1 - \partial_1 B_3] + [-\frac{\epsilon}{\tau}\frac{\partial_0 \epsilon}{\epsilon} E_2 + \frac{\partial_3 \tau}{\tau} B_1 - \frac{\partial_1 \tau}{\tau} B_3] = \frac{1}{\epsilon_0}\frac{1}{\tau} j_2, \qquad (32)$$
$$[-\frac{\epsilon}{\tau}(\partial_0 E_3) + \partial_1 B_2 - \partial_2 B_1] + [-\frac{\epsilon}{\tau}\frac{\partial_0 \epsilon}{\epsilon} E_3 + \frac{\partial_1 \tau}{\tau} B_2 - \frac{\partial_2 \tau}{\tau} B_1] = \frac{1}{\epsilon_0}\frac{1}{\tau} j_3.$$

With the use of notations:

$$\frac{1}{\epsilon}\frac{\partial\epsilon}{\partial x_1}=e_1,\ \frac{1}{\epsilon}\frac{\partial\epsilon}{\partial x_2}=e_2,\ \frac{1}{\epsilon}\frac{\partial\epsilon}{\partial x_3}=e_3,\ \frac{1}{\tau}\frac{\partial\tau}{\partial x_1}=b_1,\ \frac{1}{\tau}\frac{\partial\tau}{\partial x_2}=b_2,\ \frac{1}{\tau}\frac{\partial\tau}{\partial x_3}=b_3,\ \frac{1}{\epsilon}\frac{\partial\epsilon}{\partial x_0}=e_0,\ \frac{\epsilon}{\tau}=\Delta, \quad (33)$$

equations (32) may be written sorter:

$$\begin{aligned}
&(\partial_1+e_1(x))E_1+(\partial_2+e_2(x))E_2+(\partial_3+e_3(x))E_3=\frac{1}{\epsilon_0}\frac{1}{\epsilon}\rho,\\
&-\Delta(x)(\partial_0+e_0(x))E_1+(\partial_2+b_2(x))B_3-(\partial_3+b_3(x))B_2=\frac{1}{\epsilon_0}\frac{1}{\tau}j_1,\\
&-\Delta(x)(\partial_0+e_0(x))E_2+(\partial_3+b_3(x))B_1-(\partial_1+b_1(x))B_3=\frac{1}{\epsilon_0}\frac{1}{\tau}j_2,\\
&-\Delta(x)(\partial_0+e_0(x))E_3+(\partial_1+b_1(x))B_2-(\partial_2+b_2(x))B_1=\frac{1}{\epsilon_0}\frac{1}{\tau}j_3;
\end{aligned} \quad (34)$$

below we add equations without sources:

$$\begin{aligned}
&c\partial_0B_1+\partial_2E_3-\partial_3E_2=0,\quad +c\partial_0B_2-\partial_1E_3+\partial_3E_1=0,\\
&c\partial_0B_3+\partial_1E_2-\partial_2E_1=0,\quad \partial_1B_1+\partial_2B_2+\partial_3B_3=0.
\end{aligned} \quad (35)$$

In equations (34), the quantities $\Delta(x),\tau(x),e_0(x),e_i(x),b_i(x)$, may be associated with some effective potentials, which are determined by the tensors of electric and magnetic permittivities. In order to simplify the problem, let us impose one additional constraint *Δ(x)=1*; this leads to $\tau(x)=\epsilon(x),\quad b_i(x)=e_i(x)$. For such media, the structure of equations with sources becomes much more simple:

$$\begin{aligned}
&(\partial_1+e_1)E_1+(\partial_2+e_2)E_2+(\partial_3+e_3)E_3=\frac{1}{\epsilon_0}\frac{1}{\epsilon}\rho,\\
&-(\partial_0+e_0)E_1+(\partial_2+e_2)B_3-(\partial_3+e_3)B_2=\frac{1}{\epsilon_0}\frac{1}{\epsilon}j_1,\\
&-(\partial_0+e_0)E_2+(\partial_3+e_3)B_1-(\partial_1+e_1)B_3=\frac{1}{\epsilon_0}\frac{1}{\epsilon}j_2,\\
&-(\partial_0+e_0)E_3+(\partial_1+e_1)B_2-(\partial_2+e_2)B_1=\frac{1}{\epsilon_0}\frac{1}{\epsilon}j_3.
\end{aligned} \quad (36)$$

It should be noted that the media of the type $\tau(x)=\epsilon(x),\quad b_i(x)=e_i(x)$ related to eqs. (36) may be simulated by the Riemannian geometry [30, 31].

Now, let us consider another example: the non-uniform media anisotropic along the axis $x_3=z$, and without sources. So we impose the following constraints:

$$\epsilon_1(x)=1,\epsilon_2(x)=1,\epsilon_3(x)=\epsilon(z),\quad \tau_1(x)=1,\tau_2(x)=1,\tau_3(x)=\tau(z), \quad (37)$$

then equations with sources become simpler:

$$\begin{aligned}
&\partial_1E_1+\partial_2E_2+\partial_3E_3+\frac{1}{\epsilon_3(z)}\epsilon_3'(z)E_3=0,\quad -\frac{\epsilon}{\tau}\partial_0E_1+\partial_2B_3-\partial_3B_2-\frac{1}{\tau_3(x)}\tau_3'(z)B_2=0,\\
&-\frac{\epsilon}{\tau}\partial_0E_2+\partial_3B_1-\partial_1B_3+\frac{1}{\tau_3}\tau_3'B_1=0,\quad -\frac{\epsilon}{\tau}\partial_0E_3+\partial_1B_2-\partial_2B_1=0.
\end{aligned} \quad (38)$$

In order to simplify the problem, let us impose one additional constraint $\epsilon_3 = \tau_3 = U(z)$; for such a media, the structure of the Maxwell equations becomes much more simple:

$$\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 + \frac{U'}{U} E_3 = 0, \quad \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0,$$
$$-\partial_t E_1 + \partial_2 B_3 - \partial_3 B_2 - \frac{U'}{U} B_2 = 0, -\partial_t E_2 + \partial_3 B_1 - \partial_1 B_3 + \frac{U'}{U} B_1 = 0, -\partial_t E_3 + \partial_1 B_2 - \partial_2 B_1 = 0, \quad (39)$$
$$\partial_t B_1 + \partial_2 E_3 - \partial_3 E_2 = 0, \quad \partial_t B_2 - \partial_1 E_3 + \partial_3 E_1 = 0, \partial_t B_3 + \partial_1 E_2 - \partial_2 E_1 = 0;$$

recall that we have made the change in notations, $cB_i \to B_i$, $c\partial_0 = \partial_t$. It should be noted that the media with equations (39) may be simulated by the Riemannian geometry [30, 31].

Maxwell equations in Riemannian space-time

Maxwell equation in Minkowski space for the vacuum case:

$$(\alpha^0 \partial_0 + \alpha^j \partial_j)\Psi = J, \alpha^0 = -iI, \Psi = \begin{vmatrix} 0 \\ \boldsymbol{E} + ic\boldsymbol{B} \end{vmatrix}, J = \frac{1}{\epsilon_0} \begin{vmatrix} \rho \\ \boldsymbol{ij} \end{vmatrix} \quad (40)$$

may be extended to an arbitrary Riemannian space $dS^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta$. The new form of the main equation is based on the use of tetrad formalism, and the generalized form reads as follows:

$$\alpha^\rho(x)[\partial_\rho + A_\rho(x)]\Psi(x) = J(x),$$
$$\alpha^\rho(x) = \alpha^c e^\rho_{(c)}(x), \quad A_\rho(x) = \frac{1}{2} j^{ab} e^\beta_{(a)} \nabla_\rho e_{(n)\beta}, \quad (41)$$

where $e^\rho_{(c)}(x)$ stands for a tetrad; $A_\rho(x)$ is the so called connection; ∇_ρ stands for the covariant derivative; j^{ab} designate generators of the 3-vector representation of the complex group $SO(3.C)$, their explicit form is given in [31].

This new equation (41) takes into account the influence of the external gravitational fields on Maxwell electrodynamics. Besides, it gives tools to study the Maxwell equations with the use of any curvilinear coordinates parameterizing the space-time, in Minkowsky space as well. Equation (41) may be presented with the use of the Ricci rotation coefficients:

$$\alpha^c (e^\rho_{(c)}(x)\partial_\rho + \frac{1}{2} j^{ab} \gamma_{abc}(x))\, \Psi(x) = J(x), \quad \gamma_{bac} = -e_{(b)\beta;\alpha} e^\beta_{(a)} e^\alpha_{(c)}. \quad (42)$$

This new equations (41), (42) are symmetric under the local Lorenz group; this property correlates with the 6-parametric freedom in choosing the tetrad $e^\alpha_{(c)}(x)$ at the fixed metrical tensor $g_{\alpha\beta}(x)$.

Maxwell equations in Riemannian space in presence of media

Now we turn to the matrix form of the Maxwell equations in media, and extend it for an arbitrary Riemannian space-time. We start with the equation in Minkovski space

$$(-i\partial_0 + \alpha^i \partial_i)\, M(x) + (-i\partial_0 + \beta^i \partial_i)\, N(x) = J(x); \quad (43)$$

its generalized form, consistent with requirements of General Relativity, is

$$\alpha^{\rho}(x)(\partial_{\rho}+A_{\rho}(x))\,M(x)+\beta^{\rho}(x)(\partial_{\rho}+B_{\rho}(x))\,N(x)=J(x)\,, \tag{44}$$

two connections *A(x)* and $B_{\rho}=A^{*}(x)$ correspond to different fields variables, *M(x)* and *N(x)*.

The constitutive relations in Riemannian space-time

Let us turn again to the Maxwell equations in Riemannian space-time

$$\alpha^{\rho}(\partial_{\rho}+A_{\rho})M+\beta^{\rho}(\partial_{\rho}+B_{\rho})N=J, \tag{45}$$

and take into account the above definitions:

$$\begin{aligned}&\boldsymbol{E}+ic\boldsymbol{B}=\boldsymbol{f}\,, \qquad \frac{1}{\epsilon_0}(\boldsymbol{D}+i\boldsymbol{H}/c)=\boldsymbol{h}\,,\\ &\boldsymbol{M}=\frac{\boldsymbol{h}+\boldsymbol{f}}{2}=\frac{1}{2}(\frac{\boldsymbol{D}}{\epsilon_0}+\boldsymbol{E})+\frac{i}{2}(c\boldsymbol{B}+\frac{\boldsymbol{H}}{\epsilon_0 c})\,,\\ &\boldsymbol{N}=\frac{\boldsymbol{h}^{*}-\boldsymbol{f}^{*}}{2}=\frac{1}{2}(\frac{\boldsymbol{D}}{\epsilon_0}-\boldsymbol{E})+\frac{i}{2}(c\boldsymbol{B}-\frac{\boldsymbol{H}}{\epsilon_0 c})\,.\end{aligned} \tag{46}$$

In more detailed form, the main equation (45) reads:

$$\begin{aligned}&-i(e^{\rho}_{(0)}\partial_{\rho}+\frac{1}{2}j^{ab}\gamma_{ab0})M+\alpha^{k}(e^{\rho}_{(k)}\partial_{\rho}+\frac{1}{2}j^{ab}\gamma_{abk})M-\\ &-i(e^{\rho}_{(0)}\partial_{\rho}+\frac{1}{2}j^{ab}\gamma_{ab0})N+\beta^{k}(e^{\rho}_{(k)}\partial_{\rho}+\frac{1}{2}j^{*ab}\gamma_{abk})N=J(x)\,.\end{aligned} \tag{47}$$

Below we will use the following notations (expressions for the spin matrices s^{i} are given in the [31]):

$$\begin{aligned}&e^{\rho}_{(0)}(x)\partial_{\rho}=\partial_{(0)}, \qquad e^{\rho}_{(k)}(x)\partial_{\rho}=\partial_{(k)},\\ &\frac{1}{2}j^{ab}\gamma_{abc}=s^{1}(\gamma_{23c}+i\gamma_{01c})+s^{2}(\gamma_{31c}+i\gamma_{02c})+s^{3}(\gamma_{12c}+i\gamma_{03c})\,,\\ &\frac{1}{2}j^{*ab}\gamma_{abc}=s^{1}(\gamma_{23c}-i\gamma_{01c})+s^{2}(\gamma_{31c}-i\gamma_{02c})+s^{3}(\gamma_{12c}-i\gamma_{03c})\,,\\ &(\gamma_{01c},\gamma_{02c},\gamma_{03c})=\boldsymbol{v}_c(x), \quad (\gamma_{23c},\gamma_{31a},\gamma_{12c})=\boldsymbol{p}_c(x)\,, \quad c=0,1,2,3\,,\end{aligned} \tag{48}$$

the quantities $\boldsymbol{v}_c(x)$ and $\boldsymbol{p}_c(x)$ represent 24 Ricci rotation coefficients.Further, allowing for the constitutive relations, from (47) we get:

$$\begin{aligned}&(\alpha^{k}\partial_{(k)}+\boldsymbol{s}\boldsymbol{v}_0+\alpha^{k}\boldsymbol{s}\boldsymbol{p}_k)\frac{1}{2}\begin{vmatrix}0\\(1+\hat{\epsilon})\boldsymbol{E}\end{vmatrix}+(\partial_{(0)}+\boldsymbol{s}\boldsymbol{p}_0-\alpha^{k}\boldsymbol{s}\boldsymbol{v}_k)\frac{1}{2}\begin{vmatrix}0\\(1+\hat{\tau})\boldsymbol{B}\end{vmatrix}+\\ &+(\beta^{k}\partial_{(k)}-\boldsymbol{s}\boldsymbol{v}_0+\beta^{k}\boldsymbol{s}\boldsymbol{p}_k)\frac{1}{2}\begin{vmatrix}0\\(-1+\hat{\epsilon})\boldsymbol{E}\end{vmatrix}+(\partial_{(0)}+\boldsymbol{s}\boldsymbol{p}_0+\beta^{k}\boldsymbol{s}\boldsymbol{v}_k)\frac{1}{2}\begin{vmatrix}0\\(1-\hat{\tau})\boldsymbol{B}\end{vmatrix}=\frac{1}{\epsilon_0}\begin{vmatrix}\rho\\0\end{vmatrix},\end{aligned} \tag{49}$$

$$\begin{aligned}&(\alpha^{k}\partial_{(k)}+\boldsymbol{s}\boldsymbol{v}_0+\alpha^{k}\boldsymbol{s}\boldsymbol{p}_k)\frac{1}{2}\begin{vmatrix}0\\(1+\hat{\tau})\boldsymbol{B}\end{vmatrix}-(\partial_{(0)}+\boldsymbol{s}\boldsymbol{p}_0-\alpha^{k}\boldsymbol{s}\boldsymbol{v}_k)\frac{1}{2}\begin{vmatrix}0\\(1+\hat{\epsilon})\boldsymbol{E}\end{vmatrix}+\\ &+(\beta^{k}\partial_{(k)}-\boldsymbol{s}\boldsymbol{v}_0+\beta^{k}\boldsymbol{s}\boldsymbol{p}_k)\frac{1}{2}\begin{vmatrix}0\\(1-\hat{\tau})\boldsymbol{B}\end{vmatrix}-(\partial_{(0)}+\boldsymbol{s}\boldsymbol{p}_0+\beta^{k}\boldsymbol{s}\boldsymbol{v}_k)\frac{1}{2}\begin{vmatrix}0\\(-1+\hat{\epsilon})\boldsymbol{E}\end{vmatrix}=\frac{1}{\epsilon_0}\begin{vmatrix}0\\\boldsymbol{i}\boldsymbol{j}\end{vmatrix}.\end{aligned} \tag{50}$$

If we restrict ourselves to static metrics of the following diagonal structure (they not depend on the time variable):

$$g_{\alpha\beta}(x) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & g_{11}(x) & 0 & 0 \\ 0 & 0 & g_{22}(x) & 0 \\ 0 & 0 & 0 & g_{33}(x) \end{vmatrix}, \quad e^{\alpha}_{(a)}(x) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{1}_{(1)}(x) & 0 & 0 \\ 0 & 0 & e^{2}_{(2)}(x) & 0 \\ 0 & 0 & 0 & e^{3}_{(3)}(x) \end{vmatrix}, \quad (51)$$

we get more simple equations:

$$\begin{gathered} (e^{1}_{(1)}\partial_1 + p_2 - p_3)B_1 + (e^{2}_{(2)}\partial_2 + p_3 - p_1)B_2 + (e^{3}_{(3)}\partial_3 + p_1 - p_2)B_3 = 0, \\ -(e^{3}_{(3)}\partial_3 - p_2)E_2 + (e^{2}_{(2)}\partial_2 + p_3)E_3 - (p_3 + p_2)E_1 + \partial_0 B_1 = 0, \\ -(e^{1}_{(1)}\partial_1 - p_3)E_3 + (e^{3}_{(3)}\partial_3 + p_1)E_1 - (p_3 + p_1)E_2 + \partial_0 B_2 = 0, \\ -(e^{2}_{(2)}\partial_2 - p_1)E_1 + (e^{1}_{(1)}\partial_1 + p_2)E_2 - (p_1 + p_2)E_3 + \partial_0 B_3 = 0, \end{gathered} \quad (54)$$

$$\begin{gathered} (e^{1}_{(1)}\partial_1 + p_2 - p_3)\epsilon_1 E_1 + (e^{2}_{(2)}\partial_2 + p_3 - p_1)\epsilon_2 E_2 + (e^{3}_{(3)}\partial_3 + p_1 - p_2)\epsilon_3 E_3 = \frac{\rho}{\epsilon_0}, \\ -(e^{3}_{(3)}\partial_3 - p_2)\tau_2 B_2 + (e^{2}_{(2)}\partial_2 + p_3)\tau_3 B_3 - (p_3 + p_2)\tau_1 B_1 + \partial_0 E_1 = \frac{j_1}{\epsilon_0}, \\ -(e^{1}_{(1)}\partial_1 - p_3)\tau_3 B_3 + (e^{3}_{(3)}\partial_3 + p_1)\tau_1 B_1 - (p_3 + p_1)\tau_2 B_2 + \partial_0 E_2 = \frac{j_2}{\epsilon_0}, \\ -(e^{2}_{(2)}\partial_2 - p_1)\tau_1 B_1 + (e^{1}_{(1)}\partial_1 + p_2)\tau_2 B_2 - (p_1 + p_2)\tau_3 B_3 + \partial_0 E_3 = \frac{j_3}{\epsilon_0}. \end{gathered} \quad (55)$$

These equations are valid for any anisotropic nonuniform media, when using any curvilinear coordinates. W should remember that explicit form of the above equations depends on the choice of tetrad, and besides it depends on the used coordinates. In particular, relationships between the field functions specified for different tetrads are determined by the local gauge transformations; for more details see in [30, 31].

Conclusion

We have focused on application of the known Majorana – Oppenheimer formalism for Maxwell electrodynamics in nonuniform anisotropic media, this formalism is extended to space-time with non-Euclidian geometry. This aproach is effective when using the curvilinear coordinates in Minkowsky space as well.

In particular, we detail the case of special isotropic and nonuniform medium for which the tensor of electric and magnetic permittivity are proportional to each other; the situation is of special interest because the relevant constitutive relations may be simulated by Riemannian geometry [5, 30, 31]. We consider one simple example of such media, when it is nonuniform along the axis z. It may be noted that there are known many exact solutions of the Maxwell equations on the background of different space-time models, and each of these solutions may be considered as exact solutions in flat space but in some special media.

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Кузьмич А. М., Бурый А. В., Овсиюк Е. М. **МАТРИЧНАЯ СТРУКТУРА УРАВНЕНИЙ МАКСВЕЛЛА В НЕОДНОРОДНЫХ АНИЗОТРОПНЫХ СРЕДАХ И ГЕОМЕТРИЯ ПРОСТРАНСТВА**

Исследовано применение матричного комплексного формализма Майораны – Оппенгеймера в электродинамике Максвелла. Сначала этот подход детализирован для вакуума, далее учитывается наличие неоднородных и анизотропных сред. Выполнено обобщение ковариантного формализма на риманову геометрию пространства–времени таким образом, чтобы учесть присутствие материальных сред. Рассмотрены два простых примера: неоднородная анизотропная среда специального типа и среда, неоднородная вдоль оси z. Эти среды можно моделировать с помощью римановой геометрии.

Ключевые слова: уравнения Максвелла, формализм Майораны – Оппенгеймера, анизотропные среды, материальные уравнения, риманова геометрия, моделирование сред.